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Equilibrium Orientations of Gravity-Gradient Satellites

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THE gravitational torque that acts on an orbiting satellite represents an automatic mechanism for attitude control in space. Since configurations sufficiently elongated in the earth-vertical direction experience a positive restoring couple due to the gradient of gravitational attraction when displaced from equilibrium orientation, systems based on this principle can be of a passive character. Practical interest in gravitationally oriented satellites is conditioned by the additional amount of mechanical control which will be required to assure adequate precision of orientation. A calculation is indicated which demonstrates that there is an infinity of equilibrium satellite orientations, in which no control is therefore required. All are in the neighborhood of those discrete orientations for which the gravitational torque is zero.

An equilibrium orientation in space is defined by the directions of orbital angular velocity and instantaneous earth-vertical. If satellite principal inertia axes x_1 and x_3 occupy these directions, respectively, and x_2 is directed opposite to instantaneous linear orbital velocity, the axes form a right-handed triad, gravitational torque is zero, and orbital centrifugal forces are balanced by gravitational attraction. When small angular displacements α, β, γ are produced by rotating about x_1, x_2, x_3, respectively, the equilibrium is disturbed and the ensuing angular vibrations about satellite mass center are governed by an extended form of Euler's equations of rigid-body motion:

$$A\ddot{\alpha} + 3\Omega^2(B - C)\alpha = 0 \qquad (1)$$

$$B\ddot{\beta} + 4\Omega^2(A - C)\beta + \Omega(A - B - C)\dot{\gamma} = 0 \qquad (2)$$

$$C\ddot{\gamma} + \Omega^2(A - B)\gamma - \Omega(A - B - C)\dot{\beta} = 0 \qquad (3)$$

Principal-axis moments of inertia are denoted by A, B, C and Ω represents orbital angular speed for an orbit assumed for simplicity to be circular; dots indicate differentiations with respect to time. Eqs. (1–3) differ from the forms given in treatises on dynamics, owing to two causes.[1] One is the inclusion on the left-hand sides of the equations of the components of gravitational torque, and the other is due to the complete form of the total moment of momentum for a body in motion about a point *not* fixed in space.

In order to obtain a useful integral of the system of Eqs. (1–3), it is convenient to rewrite them as

$$\ddot{\alpha} = \partial U_1/\partial\alpha \qquad (1')$$

$$\ddot{\beta} + 2a\dot{\gamma} = \partial U_2/\partial\beta \qquad (2')$$

$$\ddot{\gamma} - 2b\dot{\beta} = \partial U_3/\partial\gamma \qquad (3')$$

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where clearly a and b are constants, and U_1, U_2, U_3 are functions only of α, β, γ, respectively, given by

$$a = \frac{\Omega(A - B - C)}{2B} \qquad b = \frac{\Omega(A - B - C)}{2C}$$

$$U_1(\alpha) = \frac{-3}{2}\Omega^2\frac{(B - C)}{A}\alpha^2 \qquad U_2(\beta) = -2\Omega^2\frac{(A - C)}{B}\beta^2$$

$$U_3(\gamma) = -\frac{1}{2}\Omega^2\frac{(A - B)}{C}\gamma^2$$

Multiplying Eqs. (1′, 2′, and 3′) in turn by $\Omega\dot{\alpha}$, $b\dot{\beta}$, and $a\dot{\gamma}$ and adding, one obtains by means of a single integration

$$(\Omega\dot{\alpha}^2 + b\dot{\beta}^2 + a\dot{\gamma}^2)/2 = \Omega U_1 + bU_2 + aU_3 + K' \qquad (4)$$

where K' is a constant of integration. An equilibrium orientation is recognized by the simultaneous and persistent conditions $\dot{\alpha} = \dot{\beta} = \dot{\gamma} = 0$. Thus if each of the angular velocities $\dot{\alpha}$, β, $\dot{\gamma}$ is put equal to zero in Eq. (4), the right-hand side furnishes the geometrical condition that is necessary to secure the corresponding equilibrium orientation. The method employed to obtain the result hardly differs from that used by Jacobi in his treatment of the problem of Lagrange's three particles.[2]

Writing now

$$U(\alpha,\beta,\gamma) = \Omega U_1 + bU_2 + aU_3$$

Eq. (4) reduces to

$$U(\alpha,\beta,\gamma) = -K' \qquad (5)$$

Although the particular orientation given by $\alpha = \beta = \gamma = 0$ is seen from Eqs. (1–3) to represent one orientation of dynamical equilibrium (that one which corresponds to zero gravitational torque), Eq. (5) shows that there is an entire one-parameter family of points in α,β,γ space for which the same condition is true. For all of the orientations, in general, a net gravitational torque is present but is exactly balanced by the couple generated by the static gradient of centrifugal forces in the orbital motion.

A case of special interest from the practical viewpoint is that in which the satellite possesses axial symmetry about the earth-vertical direction, so that $A = B$. In this case it is easy to see that Eq. (5) reduces to

$$\alpha^2 = \tfrac{2}{3}\beta^2 + K$$

and evidently the constant K must vanish, since $\alpha = 0$, $\beta = 0$ are values corresponding to one solution.

In addition to demonstrating that many equilibrium orientations are to be found in the same neighborhood of attitudes, the present result may be useful in determining advantageous contours for communications satellite reflector disks.

Finally it is noted that the present calculation bridges the gap between two well-known results. Although a spherically symmetric mass distribution possesses a double infinity of equilibrium orientations (i.e., it has no preferred directions), and two point masses attached to a weightless rod possess only six discrete equilibrium orientations (two of which are stable[3]), the calculation shows that for a general distribution of mass there are a single infinity of orientations replacing each of the six discrete ones found by Synge.

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